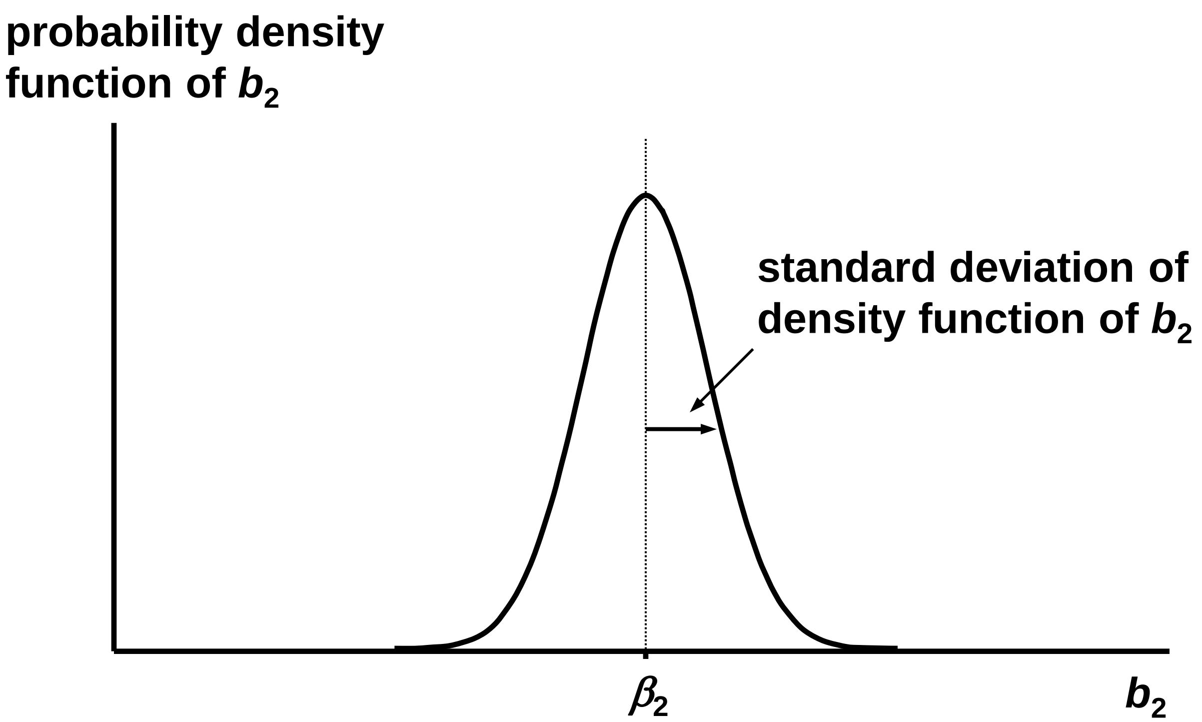
ECONOMETRICS NOTES

Unit 0 – LECTURE 4

PRECISION OF THE REGRESSION COEFFICIENTS

Simple regression model: $Y = \beta_1 + \beta_2 X + u$



We have seen that the regression coefficients b_1 and b_2 are random variables. They provide point estimates of β_1 and β_2, respectively. In the last sequence we demonstrated that these point estimates are unbiased.

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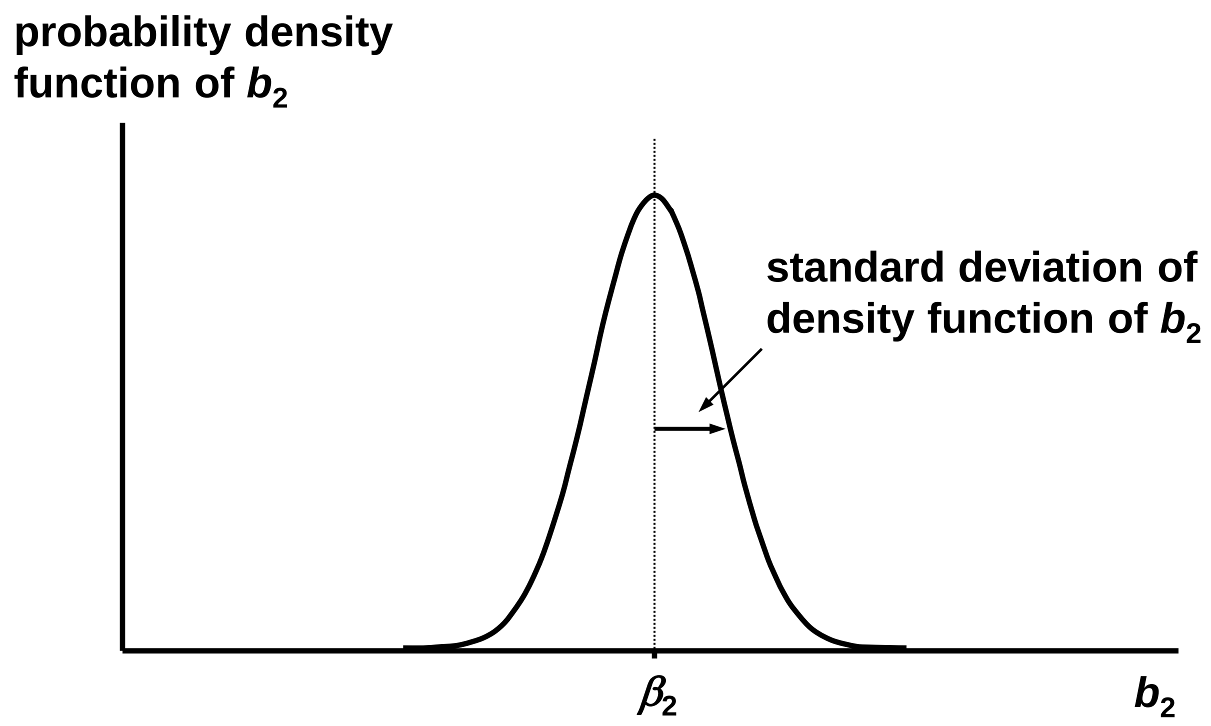

In this sequence we will see that we can also obtain estimates of the standard deviations of the distributions. These will give some idea of their likely reliability and will provide a basis for tests of hypotheses.

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$$\sigma_{b_1}^2 = \sigma_u^2 \left\{ \frac{1}{n} + \frac{\overline{X}^2}{\sum (X_i - \overline{X})^2} \right\}$$

$$\sigma_{b_2}^2 = \frac{\sigma_u^2}{\sum (X_i - \overline{X})^2}$$

Expressions (which will not be derived) for the variances of their distributions are shown above. Prove the expression for the variance of b_2.

Simple regression model: $Y = \beta_1 + \beta_2 X + u$

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We will focus on the implications of the expression for the variance of b_2. Looking at the numerator, we see that the variance of b_2 is proportional to σ_u^2. This is as we would expect, why??

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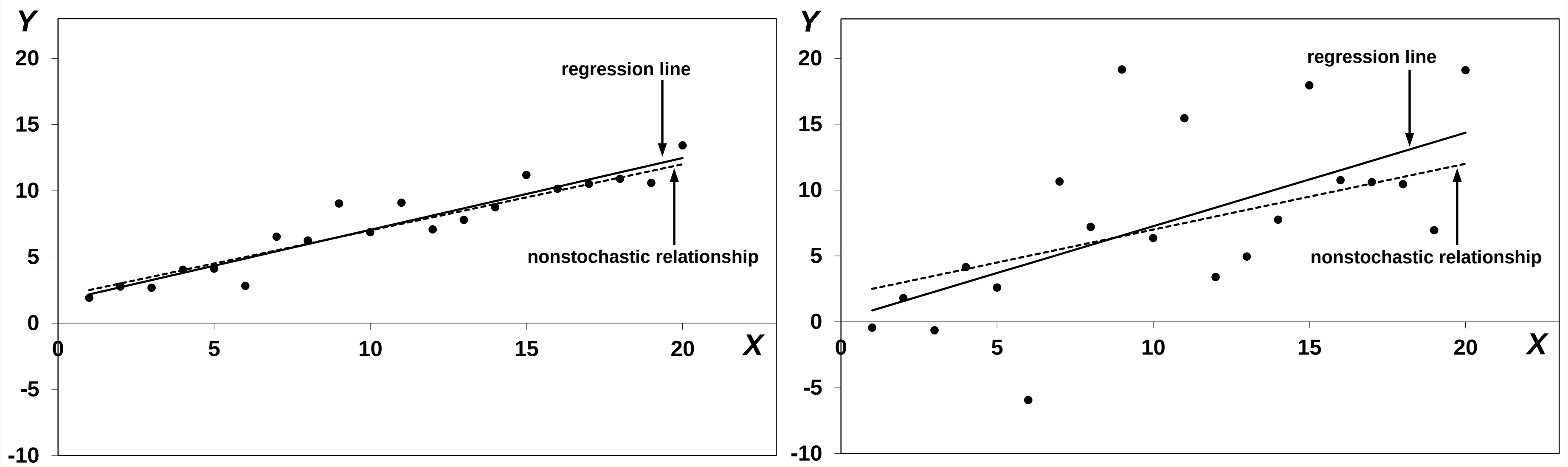
The more noise there is in the model, the less precise will be our estimates.

PRECISION OF THE REGRESSION COEFFICIENTS

Simple regression model: $Y = \beta_1 + \beta_2 X + u$

$$Y = 2.0 + 0.5X$$



This is illustrated by the diagrams above. The nonstochastic component of the relationship, $Y = 3.0 + 0.8X$, represented by the dotted line, is the same in both diagrams.

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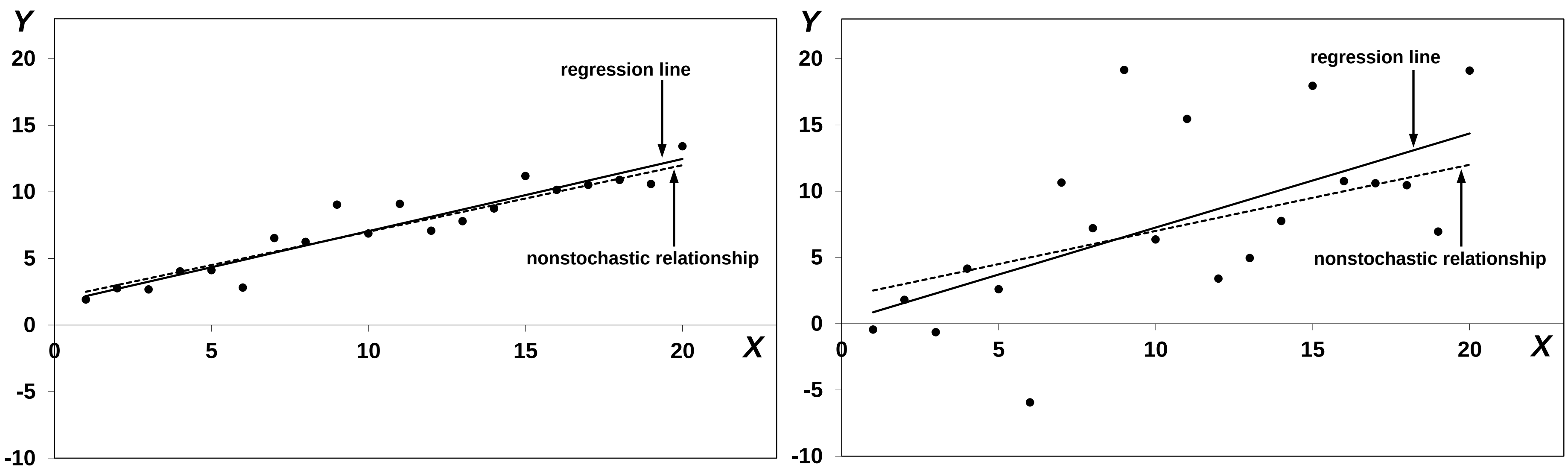

The values of *X* are the same, and the same random numbers have been used to generate the values of the disturbance term in the 20 observations.

PRECISION OF THE REGRESSION COEFFICIENTS

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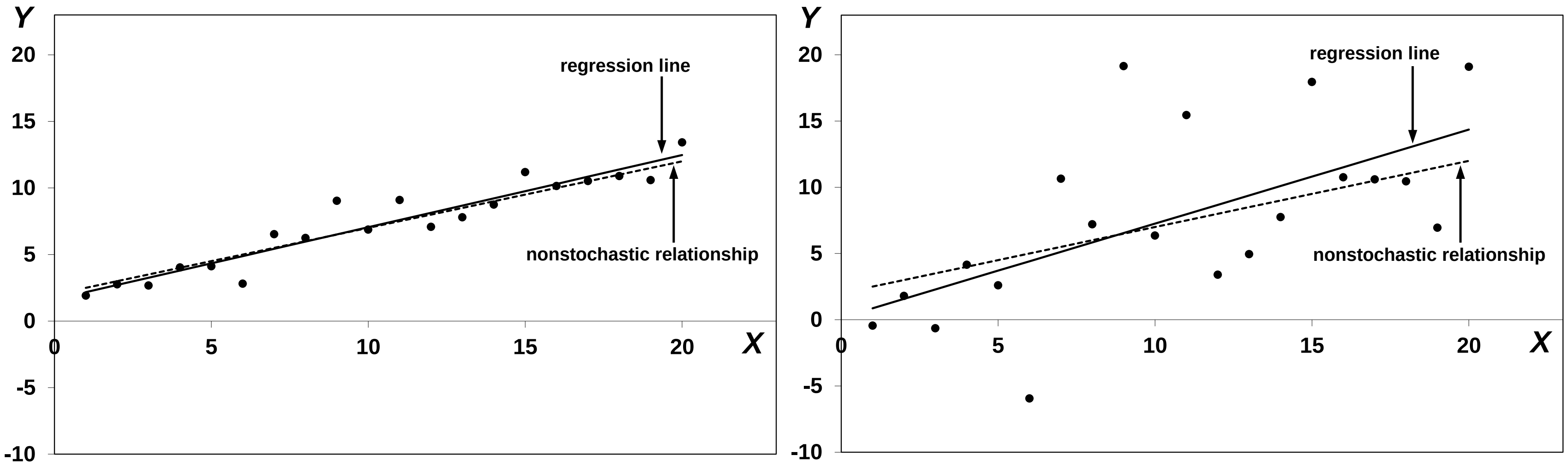

However, in the right-hand diagram the random numbers have been multiplied by a factor of 5. As a consequence, the regression line, the solid line, is a much poorer approximation to the nonstochastic relationship.

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$$\sigma_{b_1}^2 = \sigma_u^2 \left\{ \frac{1}{n} + \frac{\overline{X}^2}{\sum (X_i - \overline{X})^2} \right\}$$

$$\sigma_{b_2}^2 = \frac{\sigma_u^2}{\sum (X_i - \overline{X})^2}$$

Looking at the denominator of the expression for the variance of b_2, the larger is the sum of the squared deviations of X, the smaller is the variance of b_2.

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$$\sigma_{b_2}^2 = \frac{\sigma_u^2}{\sum (X_i - \bar{X})^2} = \frac{\sigma_u^2}{n\,\mathrm{MSD}(X)}$$

$$\mathrm{MSD}(X) = \frac{1}{n} \sum (X_i - \bar{X})^2$$

However, the size of the sum of the squared deviations depends on two factors: the number of observations, and the size of the deviations of X_i about its sample mean. To discriminate between them, it is convenient to define the mean square deviation of X, MSD(X).

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$$\sigma_{b_2}^2 = \frac{\sigma_u^2}{\sum (X_i - \bar{X})^2} = \frac{\sigma_u^2}{n \, \text{MSD}(X)}$$

$$\text{MSD}(X) = \frac{1}{n} \sum (X_i - \bar{X})^2$$

From the expression as rewritten, it can be seen that the variance of b_2 is inversely proportional to *n*, the number of observations in the sample, controlling for MSD(*X*). The more information you have, the more accurate your estimates are likely to be.

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$$\sigma_{b_2}^2 = \frac{\sigma_u^2}{\sum (X_i - \overline{X})^2} = \frac{\sigma_u^2}{n \, \mathrm{MSD}(X)}$$

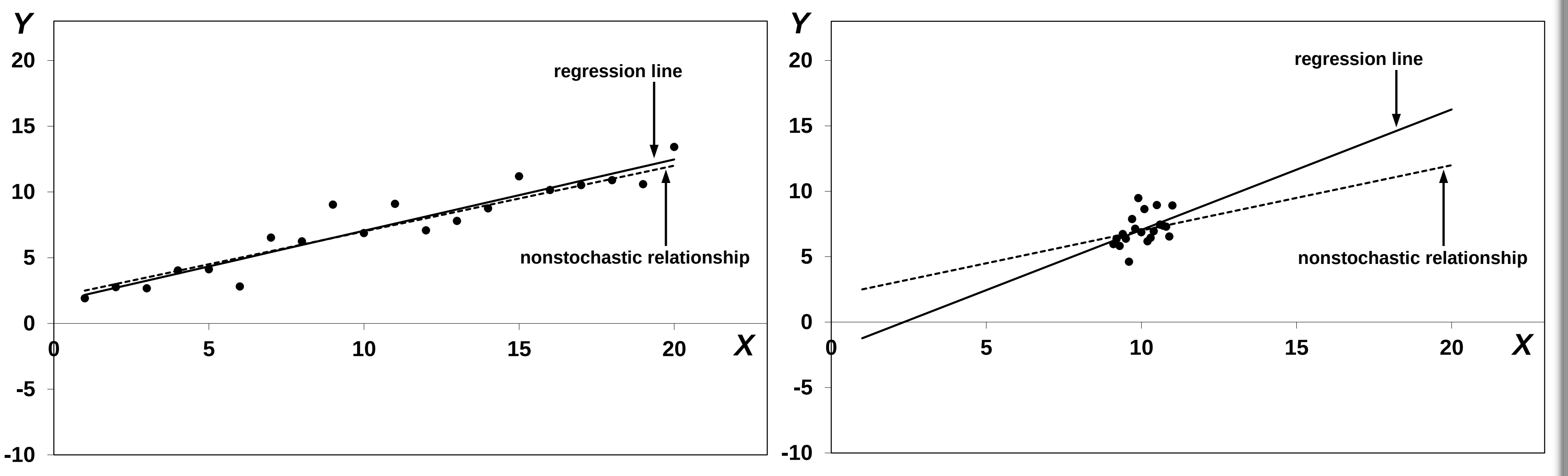
$$\mathrm{MSD}(X) = \frac{1}{n} \sum (X_i - \overline{X})^2$$

A third implication of the expression is that the variance is inversely proportional to the mean square deviation of *X*. What is the reason for this?

PRECISION OF THE REGRESSION COEFFICIENTS

Simple regression model: $Y = \beta_1 + \beta_2 X + u$

$$Y = 2.0 + 0.5X$$



In the diagrams above, the nonstochastic component of the relationship is the same and the same random numbers have been used for the 20 values of the disturbance term.

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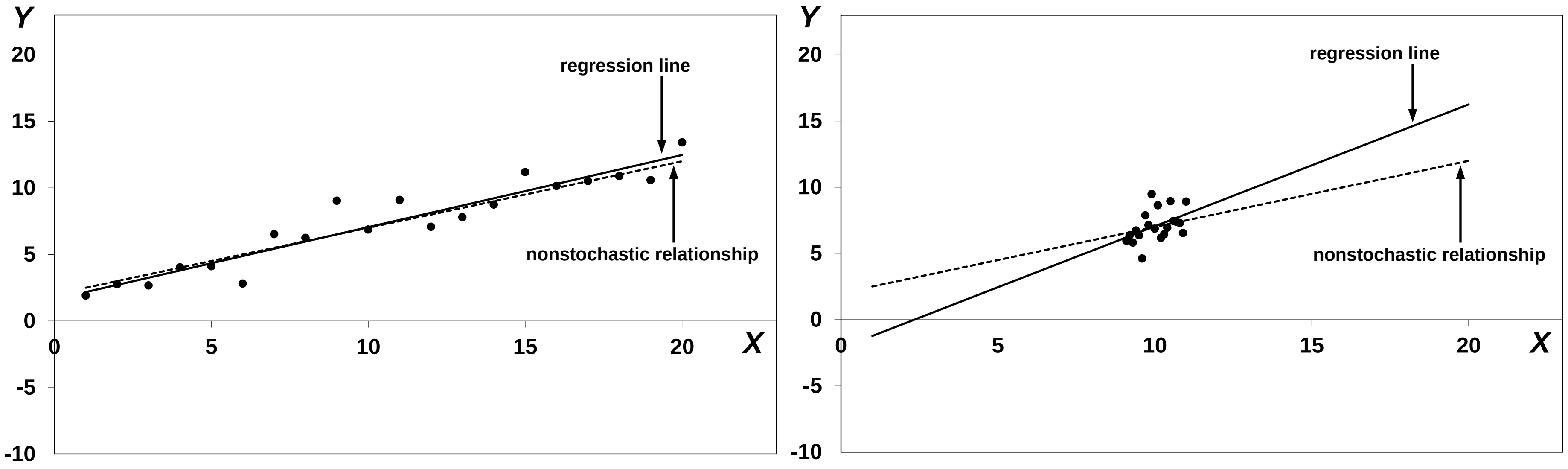
$$Y = 2.0 + 0.5X$$



However, MSD(*X*) is much smaller in the right-hand diagram because the values of *X* are much closer together.

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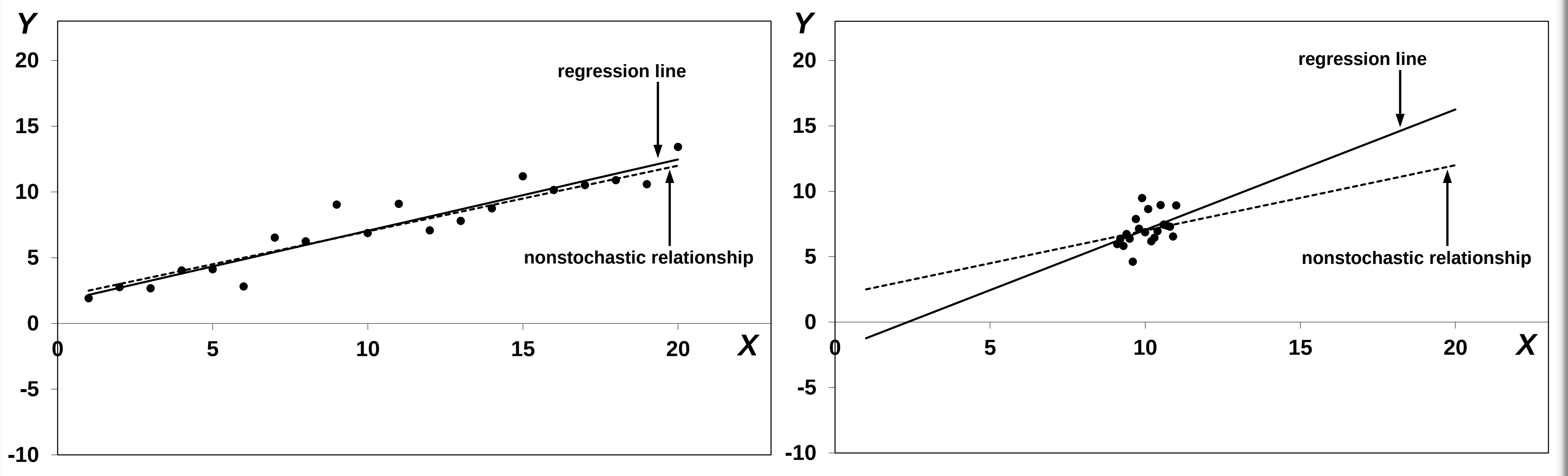

Hence, in that diagram, the position of the regression line is more sensitive to the values of the disturbance term, and as a consequence the regression line is likely to be relatively inaccurate.

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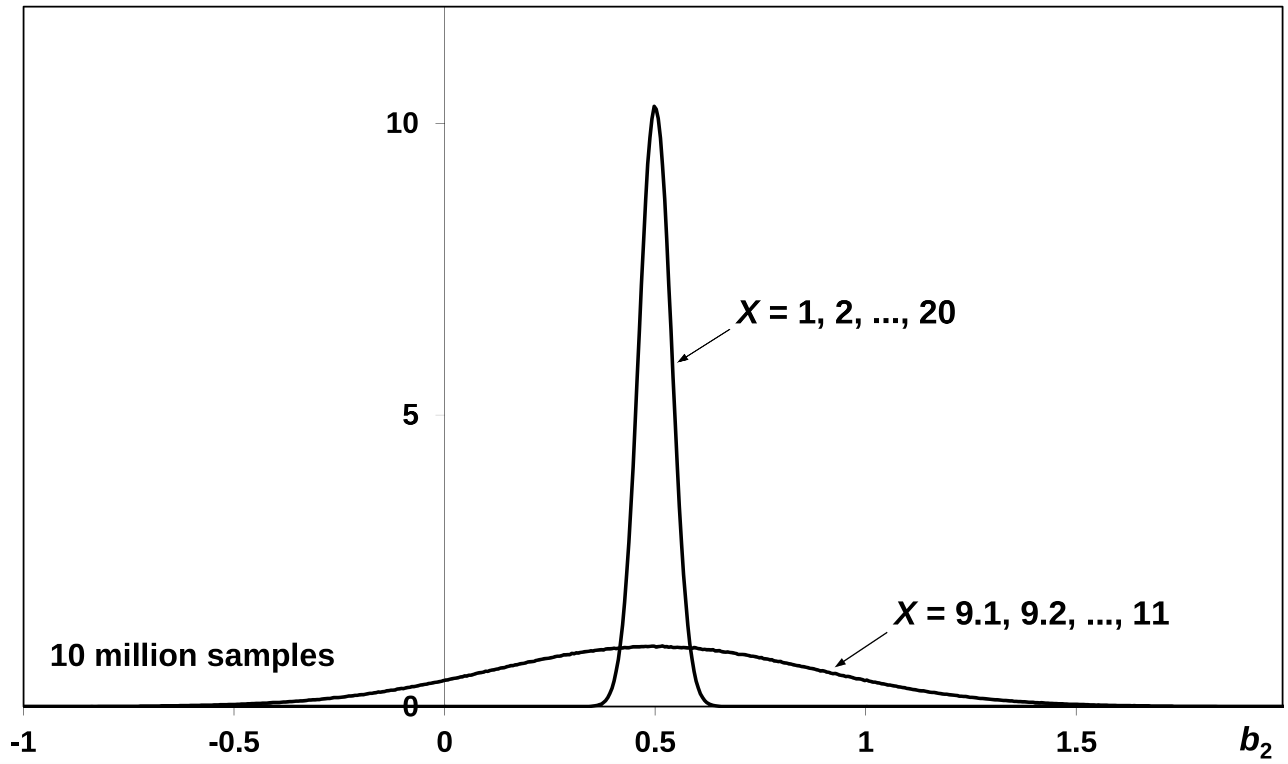
$$Y = 2.0 + 0.5X$$



The figure shows the distributions of the estimates of b_2 for X = 1, 2, ..., 20 and X = 9.1, 9.2, ..., 11 in a simulation with 10 million samples.

PRECISION OF THE REGRESSION COEFFICIENTS

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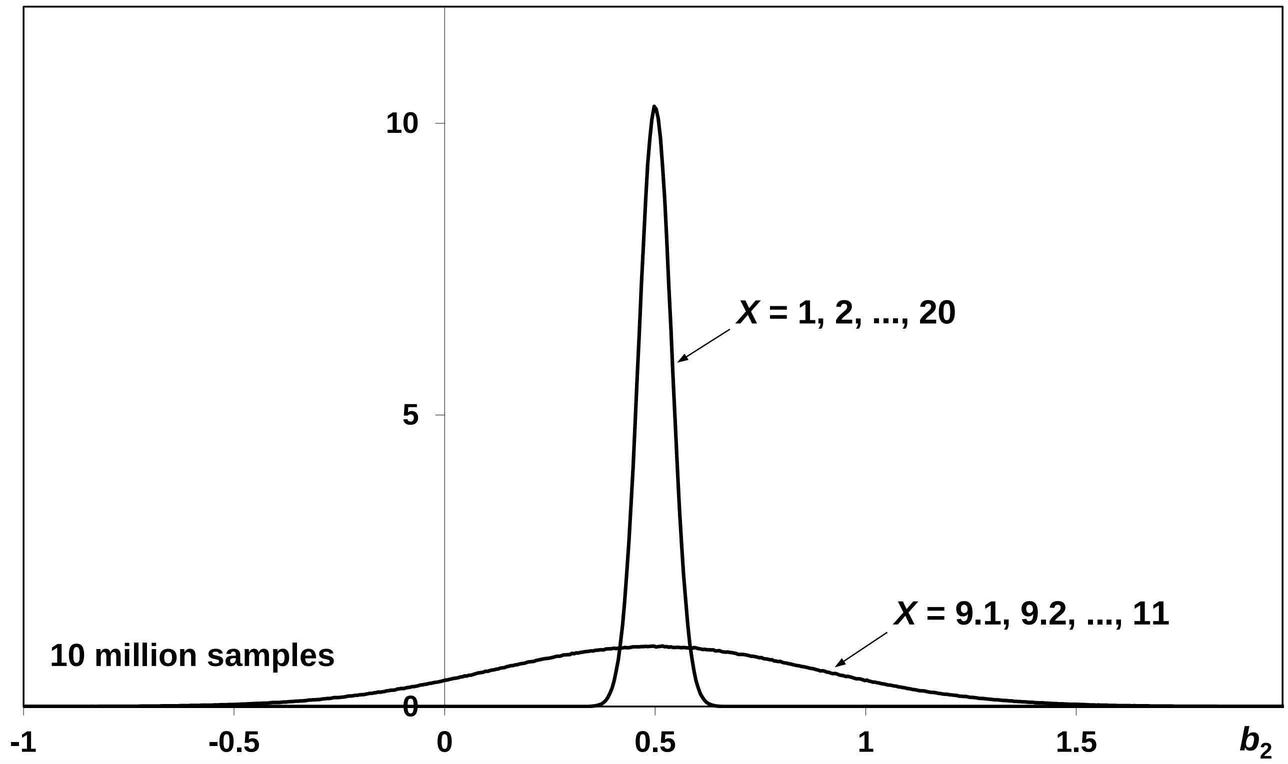

It confirms that the distribution of the estimates obtained with the high dispersion of *X* has a much smaller variance than that with the low dispersion of *X*.

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Of course, as can be seen from the variance expressions, it is really the ratio of the MSD(*X*) to the variance of *u* which is important, rather than the absolute size of either.

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We cannot calculate the variances exactly because we do not know the variance of the disturbance term. However, we can derive an estimator of σ_u^2 from the residuals.

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Clearly the scatter of the residuals around the regression line will reflect the unseen scatter of *u* about the line $Y_i = \beta_1 + b_2 X_i$, although in general the residual and the value of the disturbance term in any given observation are not equal to one another.

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$$\mathrm{MSD}(e) = \frac{1}{n} \sum (e_i - \bar{e})^2 = \frac{1}{n} \sum e_i^2$$

One measure of the scatter of the residuals is their mean square error, MSD(*e*), defined as shown. (Remember that the mean of the OLS residuals is equal to zero). Intuitively this should provide a guide to the variance of *u*.

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$$\mathrm{MSD}(e) = \frac{1}{n} \sum (e_i - \bar{e})^2 = \frac{1}{n} \sum e_i^2$$

Before going any further, ask yourself the following question. Which line is likely to be closer to the points representing the sample of observations on X and Y, the true line $Y = \beta_1 + \beta_2 X$ or the regression line $\hat{Y} = b_1 + b_2 X$?

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$$\mathrm{MSD}(e) = \frac{1}{n} \sum (e_i - \overline{e})^2 = \frac{1}{n} \sum e_i^2$$

The answer is the regression line, because by definition it is drawn in such a way as to minimize the sum of the squares of the distances between it and the observations.

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$$\mathrm{MSD}(e) = \frac{1}{n} \sum \left(e_i - \overline{e} \right)^2 = \frac{1}{n} \sum e_i^2$$

Hence the spread of the residuals will tend to be smaller than the spread of the values of *u*, and MSD(*e*) will tend to underestimate σ_u^2.

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$$\mathrm{MSD}(e) = \frac{1}{n} \sum (e_i - \overline{e})^2 = \frac{1}{n} \sum e_i^2$$

$$E(\mathrm{MSD}(e)) = \frac{n-2}{n} \sigma_u^2$$

Indeed, it can be shown that the expected value of MSD(*e*), when there is just one explanatory variable, is given by the expression above.

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$$s_u^2 = \frac{n}{n-2}\mathrm{MSD}(e) = \frac{n}{n-2}\frac{1}{n}\sum e_i^2 = \frac{1}{n-2}\sum e_i^2$$

However, it follows that we can obtain an unbiased estimator of σ_u^2 by multiplying MSD(*e*) by *n* / (*n* – 2). We will denote this s_u^2.

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$$s_u^2 = \frac{n}{n-2} \mathrm{MSD}(e) = \frac{n}{n-2} \frac{1}{n} \sum e_i^2 = \frac{1}{n-2} \sum e_i^2$$

$$\text{s.e.}(b_1) = s_u \sqrt{\frac{1}{n} + \frac{\overline{X}^2}{\sum (X_i - \overline{X})^2}} \qquad \text{s.e.}(b_2) = \sqrt{\frac{s_u^2}{\sum (X_i - \overline{X})^2}}$$

We can then obtain estimates of the standard deviations of the distributions of b_1 and b_2 by substituting s_u^2 for σ_u^2 in the variance expressions and taking the square roots.

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$$s_u^2 = \frac{n}{n-2}\mathrm{MSD}(e) = \frac{n}{n-2}\frac{1}{n}\sum e_i^2 = \frac{1}{n-2}\sum e_i^2$$

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These are described as the standard errors of b_1 and b_2, ‘estimates of the standard deviations’ being a bit of a mouthful.

PRECISION OF THE REGRESSION COEFFICIENTS

```
. reg EARNINGS S

      Source |       SS       df       MS              Number of obs =     540
-------------+------------------------------           F(  1,   538) =  112.15
       Model |  19321.5589     1  19321.5589           Prob > F      =  0.0000
    Residual |  92688.6722   538  172.283777           R-squared     =  0.1725
-------------+------------------------------           Adj R-squared =  0.1710
       Total |  112010.231   539  207.811189           Root MSE      =  13.126

------------------------------------------------------------------------------
    EARNINGS |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           S |   2.455321   .2318512    10.59   0.000     1.999876    2.910765
       _cons |  -13.93347   3.219851    -4.33   0.000    -20.25849   -7.608444
------------------------------------------------------------------------------
```

The standard errors of the coefficients always appear as part of the output of a regression. Here is the regression of hourly earnings on years of schooling discussed in a previous slideshow. The standard errors appear in a column to the right of the coefficients.

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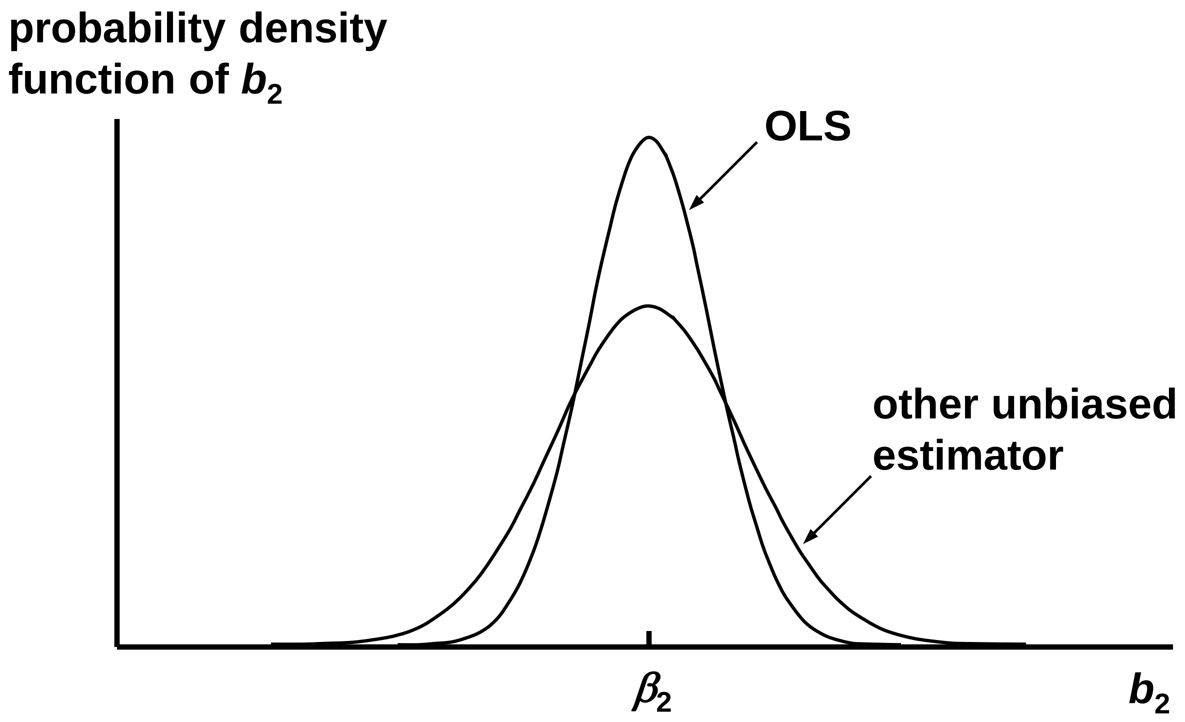

The Gauss–Markov theorem states that, provided that the regression model assumptions are valid, the OLS estimators are BLUE: best (most efficient) linear (functions of the values of *Y*) unbiased estimators of the parameters.

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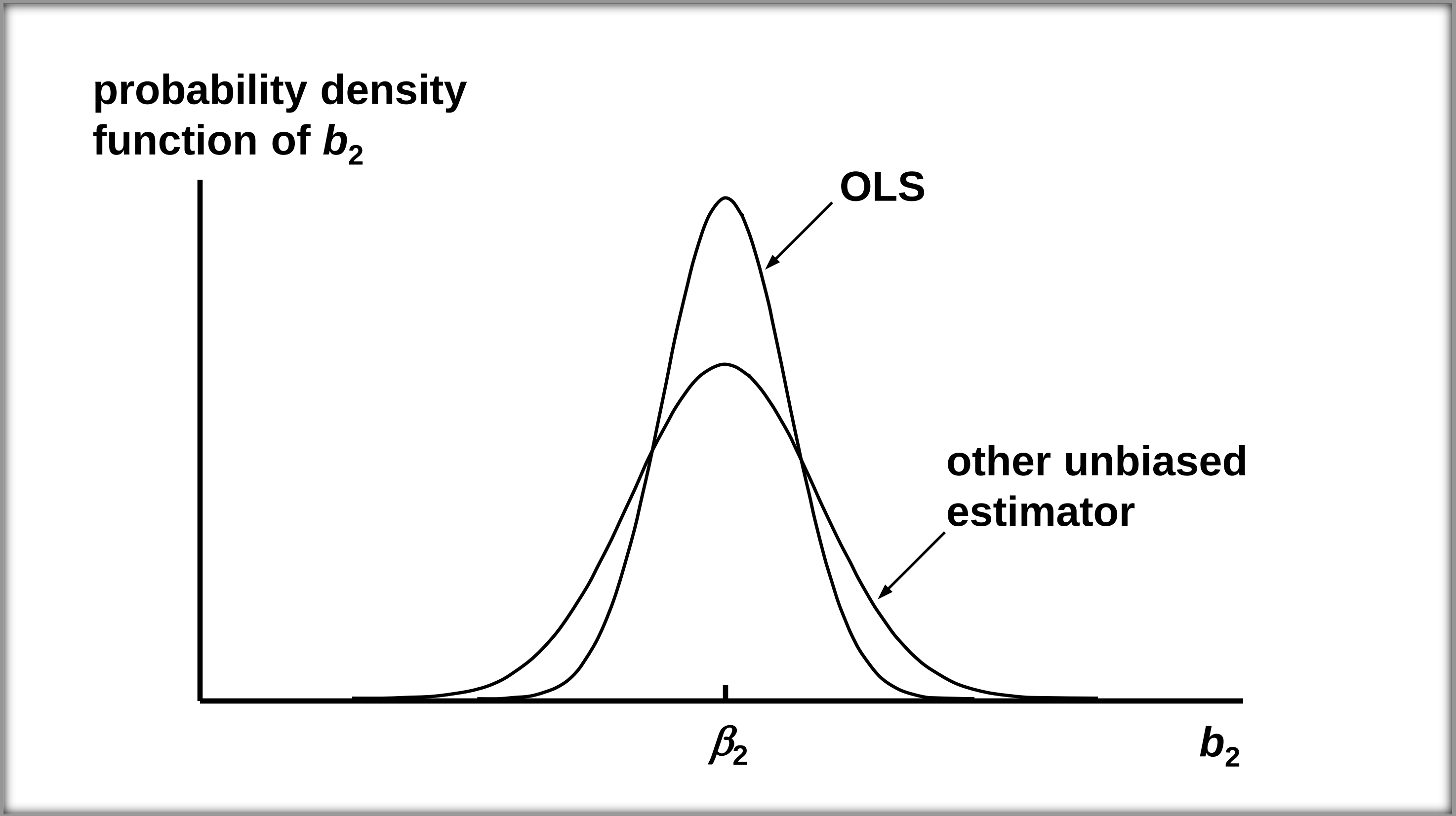

The proof of the theorem is not difficult .

Home work

Prove the Gauss–Markov theorem

END OF LECTURE